P, NP and Intractability

CS 4104: Data and Algorithm Analysis

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Virginia Tech

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Introduction

Recap: Efficient Algorithms

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- However, we've seen that some modified versions of problems we discussed don't have efficient solutions.
- Instead, the best known algorithms for these class of problems are exponential at best (if they exist).

Example: Shortest Path in a Graph

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  - Finding shortest paths in such modified graphs can make the problem NP-hard or undecidable.

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  - These constraints can turn the problem into an even more complex version that remains NP-hard and lacks a known polynomial-time solution.

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  - These modifications can turn the problem into an NP-hard problem, making it infeasible to solve in polynomial time.

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- **Role of Heuristics and Approximation Algorithms**
  - For problems that are difficult or impossible to solve exactly in polynomial time, heuristics and approximation algorithms can provide near-optimal solutions in a reasonable amount of time.
  - Understanding when and how to use these techniques is crucial for tackling real-world problems that are NP-hard.

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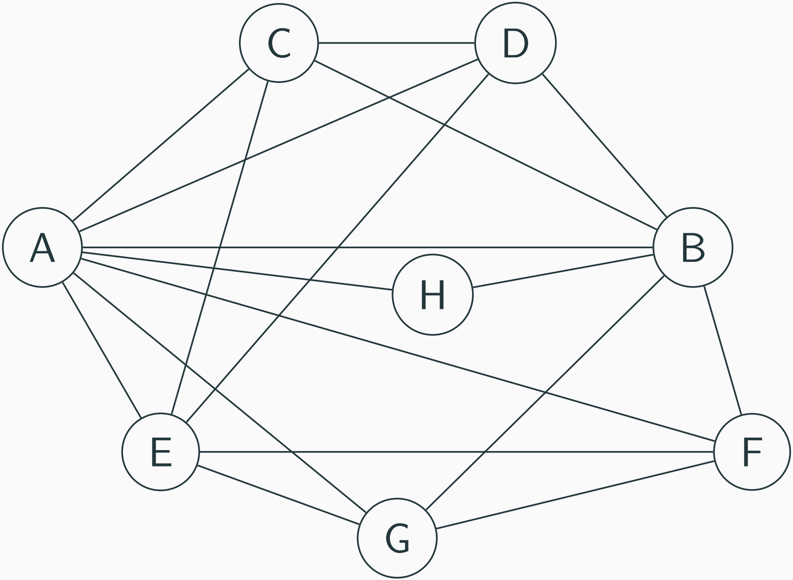
        - Which nodes and edges make such a clique ?
- No known polynomial-time solution exists for the last three
- **Brute Force Solution**: Check all possible subsets of vertices of size k to see if they form a clique, which takes exponential time.

The Clique Problem: Example



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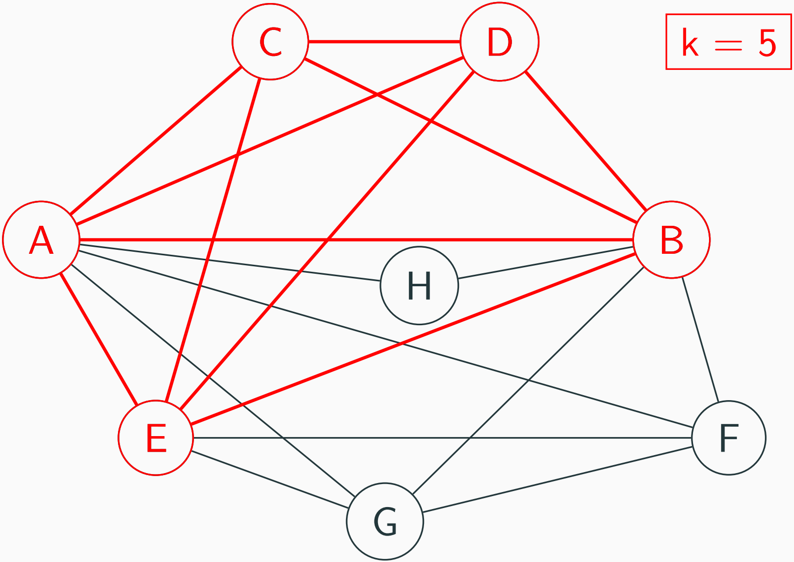

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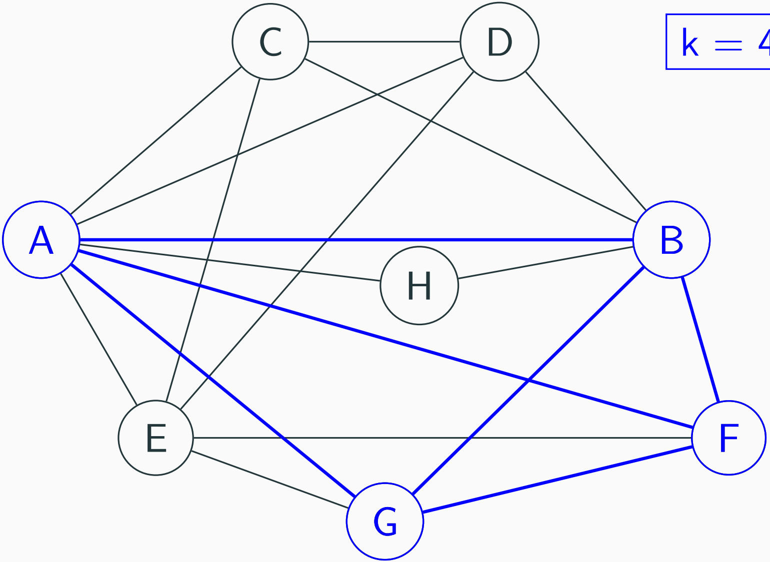

$\boxed{k = 4}$

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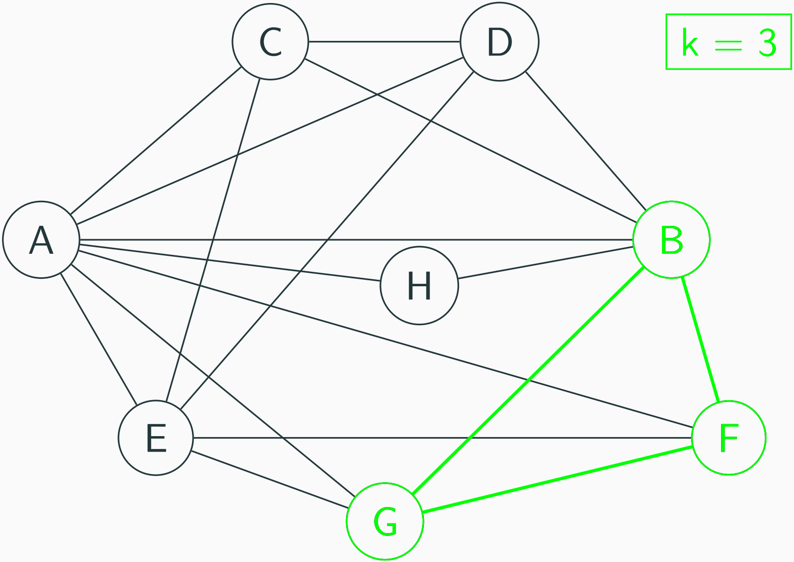

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- **Brute Force Solution**: Check all possible subsets of vertices of size k to see if they form a clique, which takes exponential time.
- Since verifying a proposed clique takes polynomial time the clique problem is NP

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  - Example: For the Hamiltonian Path problem, a certificate is a specific sequence of vertices that forms a Hamiltonian Path.
- For Co-NP problems, given a "no" answer, a certificate exists that can be checked quickly to confirm the answer is correct.
  - Example: For the Composite Number problem, a Co-NP problem, a certificate for a "no" answer (i.e., the number is prime) is a demonstration that there are no factors other than 1 and the number itself.

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    - These class of problems are *yes heavy* - we can check yes cases in polynomial time - not the no cases

NP: General Outline

```
proc solve(input):
    // This step is not well defined, hence, non-deterministic
    // If this step is done in polynomial time
    //    then we have a polynomial solution
    certificates = generateAllPossibleSolutions(input)

    // Verify the given solutions for the problem
    // This should run in polynomial time
    for certificate in certificates:
        if verifySolution(input, certificate):
            return certificate
    return "No solution found"
```

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- NP is an upper limit not a lower limit

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  - Examples: Clique Problem, Satisfiability (SAT), Hamiltonian Path, Subset Sum.

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    - If $X <_p Y$ and $Y <_p Z$, then $X <_p Z$.

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        - 3-SAT $<_p$ Hamiltonian Cycle
- A polynomial-time reduction is a process of transforming one problem into another in polynomial time.
- Polynomial-time reductions are used to show that a problem is NP-hard.
- Reductions are transitive but not symmetric.
    - If $X <_p Y$ and $Y <_p Z$, then $X <_p Z$.
    - $X <_p Y \not\Rightarrow Y <_p X$

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- A problem is said to be NP-Complete if it is in both NP and NP-hard
- It is NP in that we can verify a certificate in polynomial time
- It is NP-hard, meaning it we can reduce all NP problems to it
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- Problems in NP-complete have the property that they are difficult but if one is solved efficiently, all will be solved automatically

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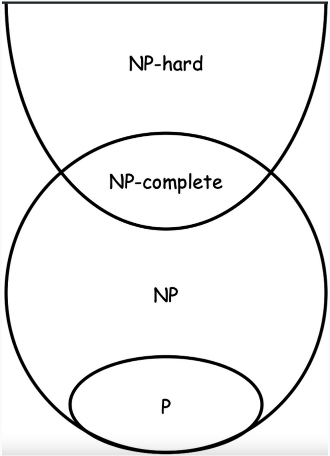

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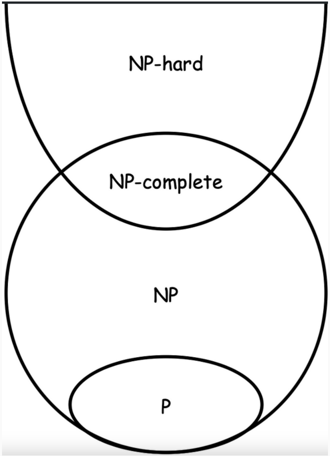

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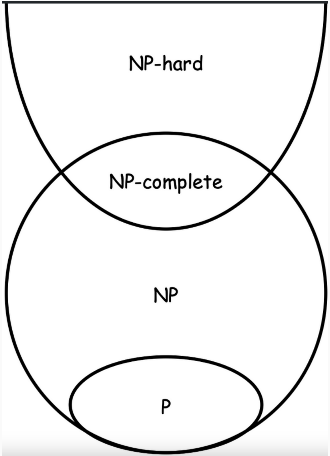

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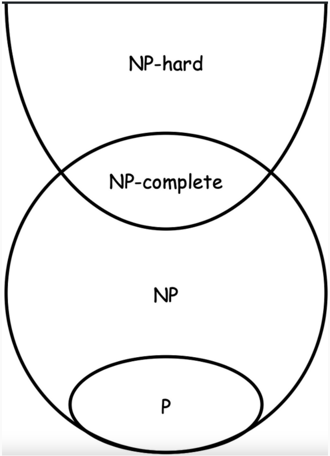

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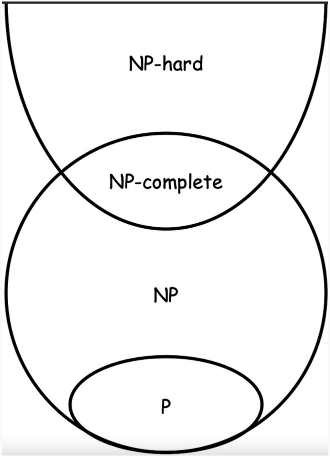

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    - Unless the problem is also NP-complete.

Image Source: https://medium.com/intuitionmath/p-np-would-mean-were-a-bunch-of-dumb-apes-20c6e50f0ba3

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        - Stop when you are left with k vertices

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- **Brute Force Solution**: Check all possible subsets of vertices of size k to see if they form an independent set, which takes exponential time.

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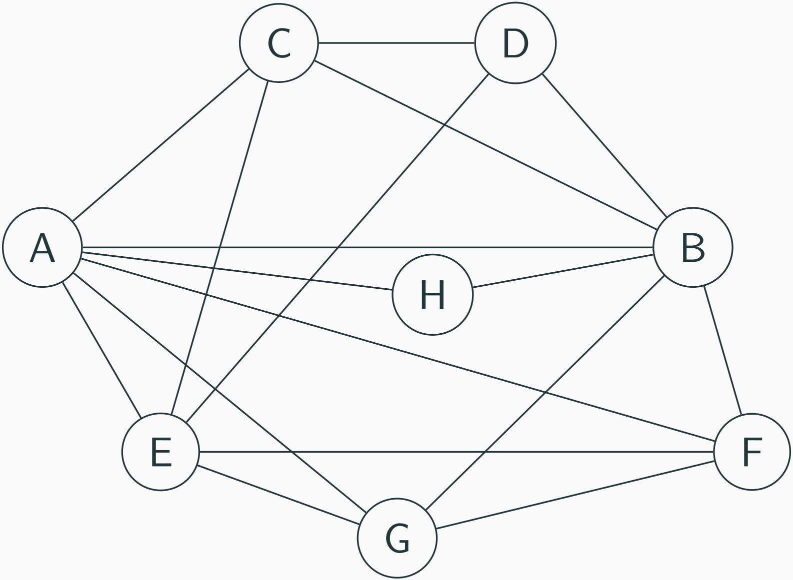

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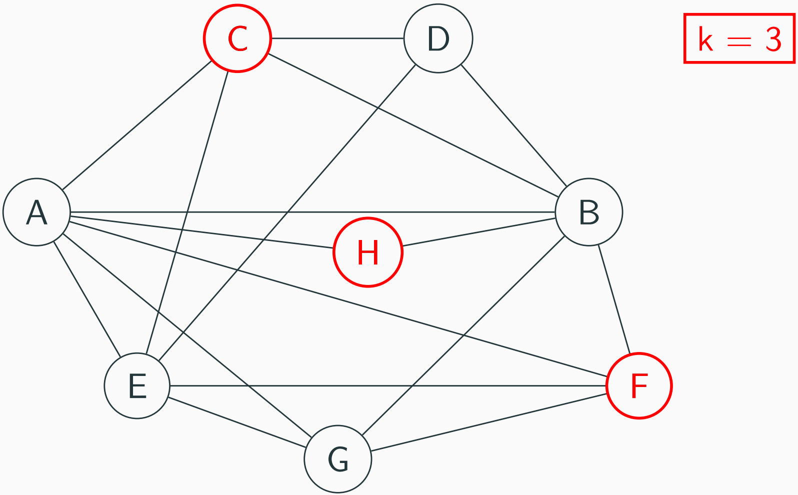

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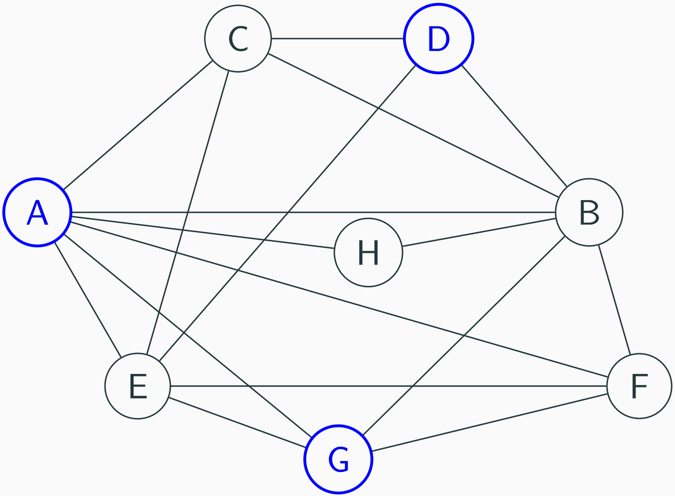

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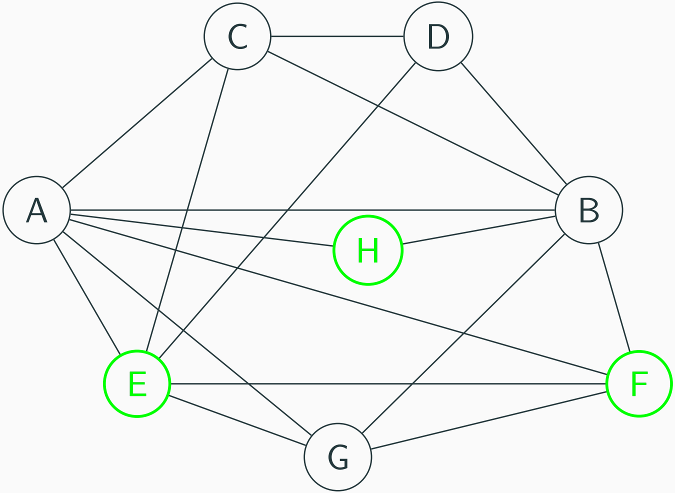

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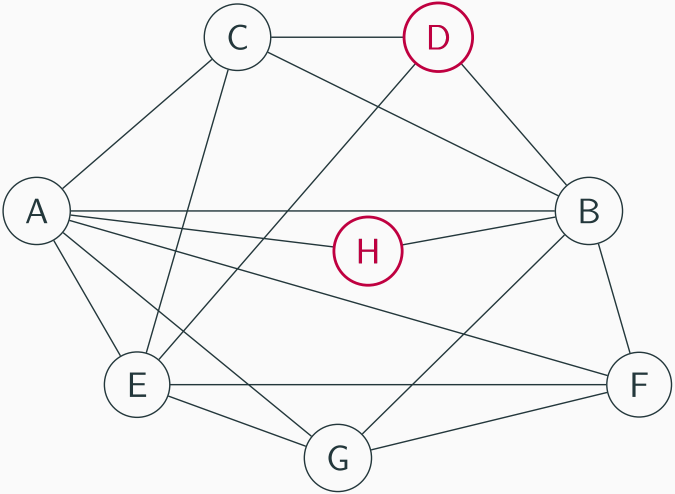

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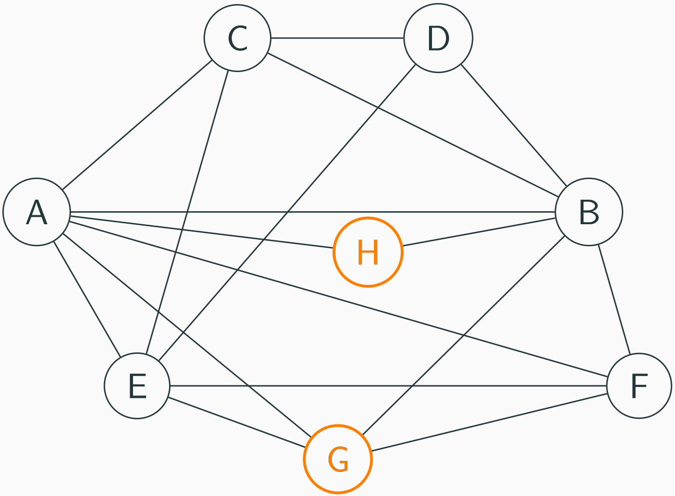

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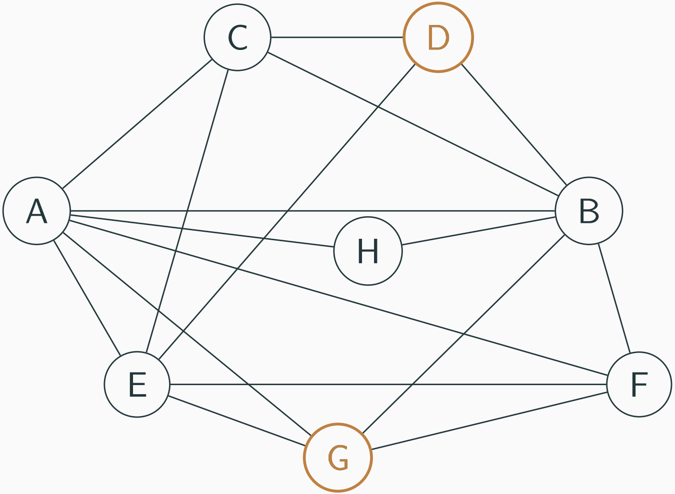

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The Independent Set Problem: NP Complete

- We want to show that the Independent Set Problem problem is NP Complete
- This means showing that it is both NP and NP hard
- To show it is NP means that if a certificate is generated for the solution we can verify it with polynomial efficiency
- To show it is NP-Hard means to show that every NP-problem can be polynomially reduced to it
    - In practice, we don't need to show reduction for every problem
    - We only need to show this for a well known NP-hard problem
    - We'll assume the clique problem is NP-hard and we show the reduction for it

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- This reduction shows that the Independent Set problem is NP-hard because the Clique problem is NP-hard.

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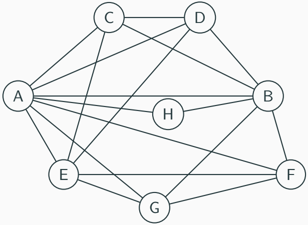
- This means our reduction is a **polynomial reduction**

Reduction: Clique to Independent Set

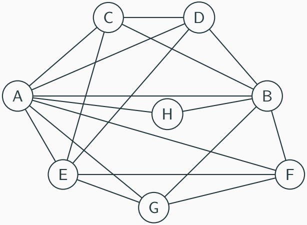
- Start with the original graph G for the Clique problem.



Graph G (Clique problem)

Reduction: Clique to Independent Set

- Start with the original graph G for the Clique problem.
- Create the complement graph G':
  - Two vertices are connected in G' if and only if they are not connected in G.



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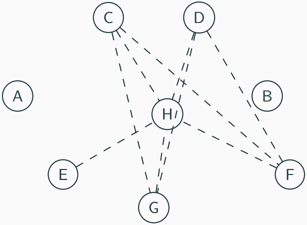

Graph G' (Independent Set problem)

Reduction: Clique to Independent Set

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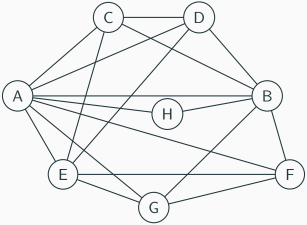
- An independent set in G' corresponds to a clique in G.



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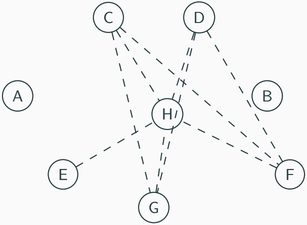
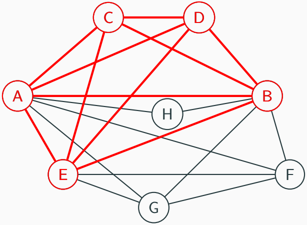


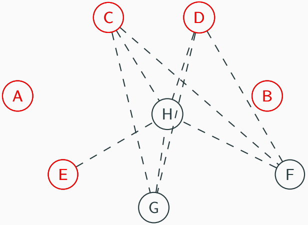
Graph G' (Independent Set problem)

Reduction: Clique to Independent Set

- Start with the original graph G for the Clique problem.
- Create the complement graph G':
  - Two vertices are connected in G' if and only if they are not connected in G.
- An independent set in G' corresponds to a clique in G.
  - For example, the independent set (A, B, C, D, E) in G' corresponds to the clique (A, B, C, D, E) in G.



Graph G (Clique problem)



Graph G' (Independent Set problem)

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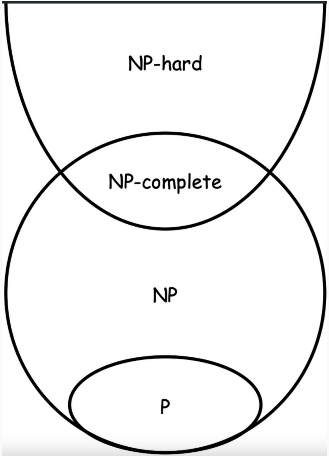
- In this particular example reduction is symmetric
  - If *clique* $<_p$ *independetset* and *independetset* $<_p$ *clique*
  - This is not the general case
    - $X <_p Y \not\Rightarrow Y <_p X$

Conclusion

P vs NP

- There are a number of questions that are still open in the P vs NP domain



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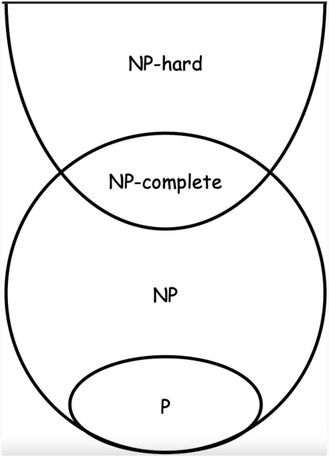

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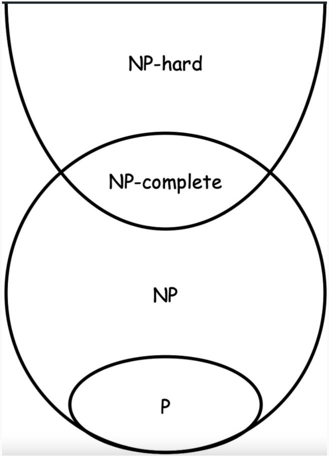

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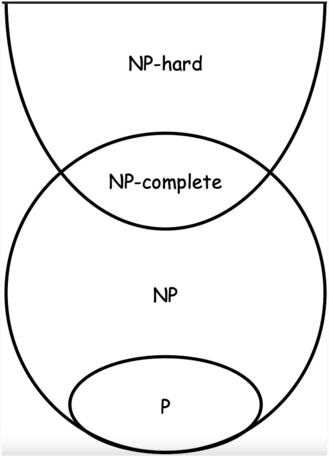

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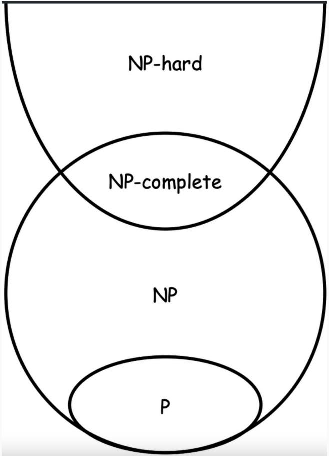

- $P = NP$?
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- $NP = co\text{-}NP$?
- $P = NP \cap co\text{-}NP$?
  - **EXP**: Verification takes exponential time
  - **PSpace** - Problems that can be verified in polynomial space given unlimited time
  - **BPP** - Problems that can be solved with probabilistic algorithm in polynomial time
  - **BQP** - Solvable via quantum computing
  - **Others**: EXSpace, 2-Exp and even unsolvable

Conclusion

- Understanding computational intractability and reductions helps classify problems and develop efficient algorithms.
- The study of NP-complete problems and their relationships with other problems is crucial in computer science.
- In practice, we use reduction to show that a problem is difficult to solve of practical purposes (i.e., intractable)
- An entire sub-field of CS/Math, called Computational complexity theory, is dedicated to studying these issues